Finding a basic feasible solution for Simplex

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1 Introduction

We show informally how to find a feasible basic solution to the simplex algorithm. The proof can be found in the text book accompanying the course (Cormen). Instead, we will describe the algorithm INITIALIZE-SIMPLEX and show how it works by example. For example, the following simple has a feasible optimal solution (what is it?):

$$\begin{aligned} \text{maximize } & x_1 + x_2 \\ \text{subject to } & x_1 + 2x_2 && \leq 18 \\ & x_1 - 3x_2 && \leq -2 \\ & x_1, x_2 && \geq 0 \end{aligned} \tag{1}$$

If we try to insert the basic solution: $x_1 = x_2 = 0$, we violate constraint (1).

When we rewrite the linear program in slack form we get:

$$\begin{aligned} z &= x_1 + x_2 \\ x_3 &= 18 - x_1 - 2x_2 \\ x_4 &= -2 - x_1 + 3x_2 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Setting $x_1 = x_2 = 0$ would mean that x_4 is negative, so the solution is infeasible. So what do we do?
Notice that for the simplex algorithm to work, we need to start with a feasible solution. For example, if we started the simplex algorithm with $x_1 = 4$ and $x_2 = 3$, that would be fine - no constraints would be violated, and we could continue from there. Likewise with any other feasible solution, like $x_1 = 10, x_2 = 4$. We now show how INITIALIZE-SIMPLEX finds a feasible basic solution.

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2 Initialize Simplex

We write the problem in slack form. Find the smallest b_i. In our example, $b_3 = 18$ and $b_4 = -2$. Label this b_ℓ. If $b_\ell \geq 0$, the basic solution is feasible, as all the constraints are met (so we can set $x_i = 0$ for all i and we have a feasible basic solution) . If it is negative, we define a new linear program as follows. We do the following (to the slack form):

1. Define a new variable, x_0.
2. Add the constraint $x_0 \geq 0$
3. Add x_0 to each contraint.
4. Set the objective function to $-x_0$.
5. PIVOT x_ℓ and x_0.
6. Run the SIMPLEX algorithm until we have an optimal solution.
7. If in the optimal solution, $x_0 = 0$ then return the slack form with x_0 removed and the original target function (This is the basic feasible solution that we then run SIMPLEX ON).
8. Otherwise, return "infeasible".

Take a look and try to understand why this works. There are two important things to notice:

1. After line (5), no constraints are negative, so we have a feasible solution to the new linear program we just defined. We can now run SIMPLEX on it and find an optimal solution.
2. Why do we set the objective function to be "*maximize* x_0"? If the optimal solution does not set $x_0 = 0$, then in fact we cannot have a feasible solution to the original problem, because that means that constraints cannot be satisfied.

3 Sample Run

We do a sample run of the Initialize Simplex algorithm on our example:

3.1 Rewrite the slack form with x_0 (stages $1-4$)

$$
\begin{aligned}
z &= -x_0 \\
x_3 &= 18 - x_1 - 2x_2 + x_0 \\
x_4 &= -2 - x_1 + 3x_2 + x_0
\end{aligned}
$$

$$x_0, x_1, x_2, x_3, x_4 \geq 0$$

3.2 Pivot x_4 and x_0 (stage 5)

$$
\begin{aligned}
z &= -2 - x_1 + 3x_2 - x_4 \\
x_3 &= 20 - 5x_2 + x_4 \\
x_0 &= 2 + x_1 - 3x_2 + x_4 \\
x_0, x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

3.3 Find the optimal solution (stages $6 - 8$)

Set $x_2 = 2/3$, and PIVOT x_2 and x_0:

$$
\begin{aligned}
z &= -x_0 \\
x_3 &= 16\frac{2}{3} + 5x_0/3 - 5x_1/3 - 2x_4/3 \\
x_2 &= 2/3 - x_0/3 + x_1/3 + x_4/3 \\
x_0, x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

Hooray! The optimal solution sets $x_0 = 0$.

3.4 What we return to SIMPLEX

Now we remove the x_0's and return the original objective function. We have:

$$
\begin{aligned}
z &= x_1 + x_2 \\
x_3 &= 16\frac{2}{3} - 5x_1/3 - 2x_4/3 \\
x_2 &= 2/3 + x_1/3 + x_4/3 \\
x_0, x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

We convert this to slack form (x_2 is neither basic nor non basic, so we have to replace it in the objective function). This gives (in slack form):

$$
\begin{aligned}
z &= 2/3 + 4x_1/3 + x_4/3 \\
x_3 &= 16\frac{2}{3} - 5x_1/3 - 2x_4/3 \\
x_2 &= 2/3 + x_1/3 + x_4/3 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

We can plug this in to the simplex algorithm.

3.5 Running the SIMPLEX algorithm with the basic feasible solution

We now run the simplex algorithm with the input: We PIVOT x_1 and x_3, and get:

$$\begin{aligned} z &= 14 - 4x_3/5 - x_4/5 \\ x_1 &= 10 - 3x_3/5 - 2x_4/5 \\ x_2 &= 4 - x_3/5 + x_4/5 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

And we are done.

We can see that setting $x_1 = 10$, $x_2 = 4$ is indeed the optimal solution!